PRINCIPLES OF ANALYSIS
LECTURE 5 - SUPREMA AND INFIMA

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1. The Real Numbers

Define the *real numbers* to be the set of all Dedekind cuts, and denote this set by $\mathbb{R}$. This is an ordered field.

For every $a \in \mathbb{Q}$, let $C_a = \{x \in \mathbb{Q} \mid x < a\}$. This is clearly a Dedekind cut; we call this the *rational cut* corresponding to a. Note that a cut is rational if and only if it is a hit.

Define a function

$$\phi : \mathbb{Q} \to \mathbb{R} \quad \text{by} \quad \phi(a) = C_a.$$

This function satisfies the following properties:

(H1) $\phi(1) = I$ (where I is the multiplicative identity in $\mathbb{R}$);
(H2) $\phi(a + b) = \phi(a) + \phi(b)$;
(H3) $\phi(ab) = \phi(a)\phi(b)$.

These properties say that ϕ is a *field homomorphism*. The image $\phi(\mathbb{Q})$ is a subfield of $\mathbb{R}$ which is isomorphic to $\mathbb{Q}$ as an ordered field. Thus we may identify the rational numbers with the set of rational cuts, and we no longer make a distinction between them. We now view $\mathbb{Q}$ as a subset of $\mathbb{R}$.

Next we note that this process has produced real numbers which did not exist in $\mathbb{Q}$; that is, the function ϕ is not surjective. To see this, we use the following exercise:

Problem 1. Let $a, b \in \mathbb{Q}$ with $0 < a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q^2 < b$.

From the rational roots theorem, we know that there is no rational number whose square is 2. However, there is a real number with this property.

Example 1. Set $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then $C \cdot C = \{x \in \mathbb{Q} \mid x < 2\} = C_2$.

Proof. First note that $1 \in C$, implying that C is a positive cut. Recall that

$$C \cdot C = \{x \in \mathbb{Q} \mid x = ab \text{ with } a, b \in C \smallsetminus M\} \cup M.$$

Let $x \in C \cdot C$. If $x \leq 0$, then $x \in C_2$, so assume that $x > 0$. Then $x = ab$ for some $a, b \in C \smallsetminus M$. Without loss of generality, assume that $b > a$. Then $x = ab \leq b^2 < 2$, so $x \in C_2$.

Let $x \in C_2$, so that $x < 2$. By the previous problem, there exists $q \in \mathbb{Q}$ such that $x < q^2 < 2$. Thus $q \in C$ so $q^2 \in C \cdot C$; since $C \cdot C$ is a cut and $x < q^2$, we must have $x \in C \cdot C$. □

Date: September 3, 2003.

Every Dedekind cut is either a hit or a gap. The image of ϕ is exactly the set of hits in the rational number line, and the irrational numbers is exactly the set of gaps.

Finally, we ask if it is possible to produce even more numbers if we repeat this process; that is, does the set of real numbers have any gaps? We demonstrate this it does not, and explore the consequences of this property.

First we need to recall a basic property of sets.

Proposition 1 (DeMorgan's Laws for Sets)**.** *Let W be a set. Let S be a subset of W and let $\mathcal{C}$ be a collection of subsets of W. Then*

(a) $S \setminus \cup_{C \in \mathcal{C}} C = \cap_{C \in \mathcal{C}} (S \setminus C)$*;*
(b) $S \setminus \cap_{C \in \mathcal{C}} C = \cup_{C \in \mathcal{C}} (S \setminus C)$*.*

Theorem 1 (Cantor-Dedekind Theorem)**.** *The set of real numbers has no gaps.*

Proof. Let $\mathcal{C}$ be a cut in $\mathbb{R}$. Let $A = \cup_{C \in \mathcal{C}} C$. Set $U = \mathbb{Q} \setminus A$. By DeMorgan's Law, $U = \cap_{C \in \mathcal{C}} C$.

Claim 1: A is a cut.

Let $a \in A$ and $u \in U$; we wish to show that $a < c$. Then $u \in \mathbb{Q} \setminus C$, for every $C \in \mathcal{C}$. But $a \in C$ for some $C \in \mathcal{C}$, and u is not in C; since C is a cut, $a < u$.

Let $a \in A$; we wish to show that a is not maximal in A. Now $a \in C$ for some $C \in \mathcal{C}$, and since C is a cut, a is not maximal in C, so there exists $c \in C$ such that $a < c$. But $C \subset A$, so $c \in A$ and a is not maximal in A.

Claim 2: $A \in \mathbb{R} \setminus \mathcal{C}$

If A were in $\mathcal{C}$, it would be the largest element in $\mathcal{C}$, because the ordering is inclusion and A contains every set in $\mathcal{C}$. In this case $\mathcal{C}$ would not be a cut.

Claim 3: A is minimal in $\mathbb{R} \setminus \mathcal{C}$

Let X be a cut, and suppose that $X < A$; we wish to show that X is in $\mathcal{C}$. Now $X < A$ means that X is strictly contained in A, so there exists $y \in A$ such that $y \notin X$. However, since $y \in A$, we know that $y \in Y$ for some $Y \in \mathcal{C}$, and we have $X < Y < A$. Since $\mathcal{C}$ is a cut, $X \in \mathcal{C}$.

This completes the proof that $\mathcal{C}$ is not a gap. □

2. Suprema and Infima

Let A be a linearly ordered set, and let $B \subset A$.

An *upper bound* for B is an element $a \in A$ such that $b \leq a$ for every $b \in B$. If B has an upper bound, we say that B is *bounded above*.

A *lower bound* for B is an element $a \in A$ such that $a \leq b$ for every $b \in B$. If B has a lower bound, we say that B is *bounded below*.

We say that B is *bounded* if it is both bounded above and bounded below.

A *supremum* of B is an element $a \in A$ such that

- **(a)** $b \leq a$ for every $b \in B$;
- **(b)** $b \leq c$ for every $b \in B$ implies $b \leq c$.

In this case, write $a = \sup(B)$.

An *infimum* of B is an element $a \in A$ such that

- **(a)** $b \geq a$ for every $b \in B$;
- **(b)** $b \geq c$ for every $b \in B$ implies $b \geq c$.

In this case write $a = \inf(B)$.

Suprema and infima are unique, if they exist. For this reason, they are sometimes referred to as *least upper bound* (lub) and *greatest lower bound* (glb), respectively.

Observation 1. Let A be a linearly ordered set. The following are equivalent conditions on A:

- **(a)** every nonempty subset of A that is bounded above has an lub;
- **(b)** every nonempty subset of A that is bounded below has a glb;
- **(c)** A has no gaps.

A linearly ordered set satisfying any one of these equivalent conditions is called *complete*.

Proposition 2. *The real numbers are a complete ordered field.*

Henceforth, we prove all that we need using this characterization of the real numbers.

3. Density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$

Let $\mathbb{R}$ be the set of real numbers. View the $\mathbb{Q}$ denote the set of rationals cuts, which we now refer to as rational numbers. Set $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$; this is the set of *irrational numbers.*

Proposition 3 (Archimedean Property)**.** *Let $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$. Then there exists $n \in \mathbb{N}$ such that $b \leq na$.*

Proof. Suppose that the Archimedean property fails. Then there exists $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ such that $na < b$ for every $n \in \mathbb{N}$. Set $A = \{na \mid n \in \mathbb{N}\}$. Now A is bounded above by b, so by the completeness property of $\mathbb{R}$, there exists $s \in \mathbb{R}$ such that $s = \sup(A)$. Since $a > 0$ we have $s < s + a$, so $s - a < s$. Since s is a least upper bound for A, $s - a$ is not an upper bound for A; thus there exists $na \in A$ such that $s - a < na$. This implies that $s < (n + 1)a \in A$, which contradicts that $s = \sup(A)$. □

Proposition 4 (Density of $\mathbb{Q}$)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Set $c = b - a$, and note that $c > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $1 \leq nc$, which shows that $1 \leq nb - na$, or $na + 1 \leq nb$.

Let $m = \min\{x \in \mathbb{N} \mid na < x\}$; this m exists by the Well-Ordering Principle of the natural numbers. Now $m \leq na + 1$, for otherwise $na + 1 < m$ and $na < m - 1 < m$, contradicting the minimality of m. Therefore $na < m < na + 1 < nb$. Divide by n to achieve $a < \frac{m}{n} < b$. With $q = \frac{m}{n}$, the proof is complete. □

Proposition 5 (Density of $\mathbb{I}$)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists $x \in \mathbb{I}$ such that $a < x < b$.*

Proof. First observe that if $q \in \mathbb{Q}$ and $x \in \mathbb{I}$, then $q + x \in \mathbb{I}$.

Let $q \in \mathbb{Q}$ be a rational number such that $a - x < q < b - x$. Then $a < q + x < b$, with $q + x \in \mathbb{I}$. □

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